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Target space equivariant cohomological structure of the Poisson sigma model

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Abstract

We study a formulation of the standard Poisson sigma model in which the target space Poisson manifold carries the Hamilton action of some finite-dimensional Lie algebra. We show that the structure of the action and the properties of the gauge invariant observables can be understood in terms of the associated target space equivariant cohomology. We use a de Rham superfield formalism to efficiently explore the implications of the Batalin–Vilkoviski master equation.

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1. Introduction

A Poisson manifold is a manifold equipped with a Poisson structure. The Poisson sigma model associates to any Poisson manifold a two-dimensional sigma model having the Poisson manifold as target space [1–3]. By means of suitable choices of the Poisson structure, it is possible to reproduce a wealth of interesting models, such as two-dimensional R^2 gravity, two-dimensional gauge theory and two-dimensional Wess–Zumino–Witten model (see Ref. [4] for a clear review). More recently, Kontsevich's formulation of deformation quantization of the algebra of functions on a Poisson manifold [5] has been interpreted in terms of the perturbation theory of the corresponding quantum Poisson sigma model on the

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two-dimensional disk [6–9]. This universality and versatility of the Poisson sigma model justifies the large body of literature devoted to its study.

In many respects, the Poisson sigma model is a gauge theory whose symmetry is based on a Poisson algebra. This makes it interesting, but it also poses a number of new problems, especially at the quantum level, due to the singularity of the kinetic terms and the nonlinearity of the symmetry. In this respect, the Batalin–Vilkoviski quantization algorithm [10] is essential for achieving a consistent quantization [6,9]. We feel that a more thorough investigation of the geometry of the model is desirable to reach a better handle on these issues.

It is often stated that the Poisson sigma model is a kind of topological field theory. The known topological field theories are divided into two broad classes, those of Schwartz or metric independent type, such as the three-dimensional Chern–Simons theory and the two-dimensional BF theory, and those of Witten or cohomological type, such as supersymmetric quantum mechanics and the Donaldson–Witten theory (see [11,12] for a detailed treatment of these models and exhaustive referencing). The Poisson sigma model does not seem to fall in either of them. However, we have found that, when the target space Poisson manifold carries the Hamilton action of some finite-dimensional Lie algebra, as it happens in virtually all the most interesting examples, it has a hidden equivariant cohomological structure, that makes it akin to the cohomological theories and determines to a considerable degree the structure of the action and the properties of the gauge invariant observables. The present paper is devoted to the study of this matter.

Our analysis relies to a great extent on an abstract algebraic framework, called operation, whose main properties are reviewed in Section 2. Briefly, a $\mathfrak{g}$ operation over Z consists of a Lie algebra $\mathfrak{g}$, a graded associative algebra Z and a set of derivations $j(\xi)$, $l(\xi)$, $\xi \in \mathfrak{g}$, and s on Z of degrees $-1, 0, +1$, respectively, satisfying the graded commutation relations (1.1). Every $\mathfrak{g}$ operation over Z admits a canonical $\mathfrak{g}$ equivariant extension. There are three cohomologies associated with the nilpotent derivation s: ordinary, $\mathfrak{g}$ basic and $\mathfrak{g}$ equivariant.

The elements of our construction are provided by the rich geometry of Poisson manifolds, whose basic facts are reviewed in Section 3. The main geometric datum of a Poisson manifold M is a 2-vector ϖ^{ij}, satisfying the Poisson condition (2.1), in terms of which the Poisson brackets $\{,\}$ on the algebra of functions on M are defined. The Lie algebra of Poisson vector fields of M is the symmetry Lie algebra of the Poisson structure of M and thus plays an essential role. In applications, however, it is often natural to restrict oneself to the Lie subalgebra of Hamilton vector fields.

When a Poisson manifold M carries the Poisson or Hamilton action of some Lie algebra $\mathfrak{h}$, one can define an $\mathfrak{h}$ operation over the space of functions of the superbundle $\Pi T \Pi T^* M$ and its $\mathfrak{h}$ equivariant extension. $\Pi T \Pi T^* M$ allows for a natural unified description of the induced $\mathfrak{h}$ action on multivectors and forms of M. This construction is expounded in detail in Sections 4–8.

Forms on a two-dimensional manifold Σ can be viewed as elements of the space of functions on superbundle $\Pi T \Sigma$, which we shall call de Rham superfields. This formalism, which is illustrated in Section 9, turns out to be elegant and convenient.

One can construct a de Rham superfield realization of the $\mathfrak{h}$ equivariant operation over $\Pi T \Pi T^* M$ simply by promoting each of its generators x^i, y_i, γ^a, etc. to a de Rham superfield. This leads to an $\mathfrak{h}$ operation over a formal graded associative algebra of superfields, referred to as $\mathfrak{h}$ Hamilton de Rham superfield operation. The $\mathfrak{h}$ basic cohomology of this is intimately related to the $\mathfrak{h}$ equivariant cohomology of $\Pi T \Pi T^* M$. Although obvious, this fact lies at the heart of our analysis of the Poisson sigma model, expounded in Sections 10–12, which we now outline briefly.

Consider a Poisson manifold M with Poisson 2-vector ϖ^{ij} carrying the Hamilton action of some finite-dimensional Lie algebra $\mathfrak{h}$. Let h_a be the functions of M corresponding via the action to fiducial generators t_a of $\mathfrak{h}$ with structure constants c_{ab}^c, so that

$$\{h_a, h_b\} = c_{ab}^c h_c. \tag{1.1}$$

Finally, let π^{ij} be a 2-vector of M. The action of the Poisson sigma model is

$$\mathcal{S}_\pi = \int_\Sigma \mu \left(y_i \, \mathrm{d}x^i + \frac{1}{2}\pi^{ij}(x) y_i y_j - \mathrm{d}\gamma^a h_a(x) \right), \tag{1.2}$$

where x^i, y_i, γ^a are generators of the $\mathfrak{h}$ Hamilton de Rham superfield operation and μ the integration supermeasure of Σ. $\mathcal{S}_\pi$ satisfies the Batalin–Vilkoviski classical master equation if π^{ij} is a Poisson 2-vector and if

$$\pi^{ij}\partial_j h_a = 0. \tag{1.3}$$

So, π^{ij} defines another Poisson structure on M with respect to which the functions h_a are Casimir (cf. Eq. (2.5)). However, we stress that, when we refer to M as a Poisson manifold, the underlying Poisson structure implied is that associated to ϖ^{ij}. The crucial result is

$$j(r)\mathcal{S}_\pi = 0, \qquad l(r)\mathcal{S}_\pi = 0, \qquad s\mathcal{S}_\pi = 0 \tag{1.4}$$

for $r \in \mathfrak{h}$, if π^{ij} Schouten commutes with ϖ^{ij}. Eq. (1.4) then states that $\mathcal{S}_\pi$ is a representative of a degree 0 $\mathfrak{h}$ basic cohomology class of Hamilton de Rham superfield operation. The Batalin–Vilkoviski nilpotent variation δ_π and the derivations $j(r)$, $l(r)$ and s (anti)commute. Thus, the Batalin–Vilkoviski cohomology and the $\mathfrak{h}$ basic Hamilton de Rham superfield cohomology are compatible.

When $\pi^{ij} = \varpi^{ij}$ the $\mathfrak{h}$ equivariant cohomology of $\Pi T \Pi T^* M$ degenerates in the ordinary one, because, by (1.3), the Hamilton action is given through Casimir functions and thus is trivial. Correspondingly, the $\mathfrak{h}$ basic Hamilton de Rham superfield cohomology reduces to the Batalin–Vilkoviski cohomology.

This is not as disappointing as it may seem at first sight. It often happens that a complicated Poisson 2-vector π^{ij} can be written as

$$\pi^{ij} = \varpi^{ij} + \vartheta^{ij}, \tag{1.5}$$

where ϖ^{ij} is a well understood Poisson 2-vector and ϑ^{ij} another Poisson 2-vector Schouten commuting with ϖ^{ij}. One can then view π^{ij} as a perturbation of ϖ^{ij} and try to understand the π^{ij} Poisson sigma model as a perturbation of the ϖ^{ij} Poisson sigma model. There are plenty of such examples, such as the R^2 gravity Poisson sigma model and the affine Lie–Poisson sigma model. More examples are illustrated in Section 13.

All known topological field theories of cohomological type are characterized by a Lie group $\mathcal{G}$, a space of fields $\mathcal{P}$ carrying a right $\mathcal{G}$ action, a certain $\mathcal{G}$ operation over $\mathcal{P}$ and an action S that is a representative of a degree 0 $\mathcal{G}$ basic or equivariant cohomology class of $\mathcal{P}$ [13]. Apparently, the Poisson sigma model has a very similar structure. The analogy, however, stops here. In the topological field theories of cohomological type, the group $\mathcal{G}$ is infinite-dimensional and the action S is a Mathai–Quillen representative of the Thom class of some vector bundle $\mathcal{E}$ over $\mathcal{P}/\mathcal{G}$ and describes localization on the zero locus of certain sections of $\mathcal{E}$ [14,15]. So far, a similar interpretation does not seem to be possible for the version of the Poisson sigma model studied in this paper.

2. Generalities on basic and equivariant cohomology

Our analysis of the Poisson sigma model is based on a formal algebraic framework, called operation, whose main properties we shall now review (see Ref. [16] for background material).

A $\mathfrak{g}$ operation over Z is a quintuplet $(Z, \mathfrak{g}, j, l, s)$, where $\mathfrak{g}$ is a Lie algebra, Z a graded associative algebra and $j(\xi)$, $l(\xi)$, $\xi \in \mathfrak{g}$, and s the graded derivations on Z of degree -1, 0, $+1$, respectively, satisfying Cartan's algebra:

$$
\begin{aligned}
&[j(\xi), j(\eta)] = 0, \qquad [l(\xi), j(\eta)] = j([\xi, \eta]), \qquad [l(\xi), l(\eta)] = l([\xi, \eta]),\\
&[s, j(\eta)] = l(\eta), \qquad [s, l(\eta)] = 0, \qquad [s, s] = 0,
\end{aligned}
\tag{2.1}
$$

where the above are graded commutators. In all examples considered below, the graded algebra Z is finitely generated. So, the graded derivations $j(\xi)$, $l(\xi)$ and s are completely defined by their action on a suitable set of natural generators.

Since $s^2 = 0$, one can define the cohomology of the differential complex (Z, s). This is called ordinary cohomology of Z. More importantly, one may consider the differential complex (Z_{basic}, s), where Z_{basic} is the s invariant subalgebra of Z annihilated by all $j(\xi)$, $l(\xi)$, $\xi \in \mathfrak{g}$. The corresponding cohomology is referred to as $\mathfrak{g}$ basic cohomology of Z.

To any Lie algebra $\mathfrak{g}$, there is canonically associated the Weil operation $(W(\mathfrak{g}), \mathfrak{g}, j, l, s)$. Here, $W(\mathfrak{g}) = A(\mathfrak{g}^\vee) \otimes S(\mathfrak{g}^\vee)$, where $A(\mathfrak{g}^\vee)$ and $S(\mathfrak{g}^\vee)$ are the antisymmetric, symmetric algebras of $\mathfrak{g}^\vee$, respectively. The natural $\mathfrak{g}$-valued generators ω, Ω of $W(\mathfrak{g})$ carry degrees 1, 2, respectively, and satisfy

$$
\begin{aligned}
&j(\xi)\omega = \xi, \qquad j(\xi)\Omega = 0, \qquad l(\xi)\omega = -[\xi, \omega], \qquad l(\xi)\Omega = -[\xi, \Omega],\\
&s\omega = \Omega - \tfrac{1}{2}[\omega, \omega], \qquad s\Omega = -[\omega, \Omega].
\end{aligned}
\tag{2.2}
$$

The cohomology of $W(\mathfrak{g})$ is trivial. The $\mathfrak{g}$ basic cohomology of $W(\mathfrak{g})$ is isomorphic to the ad $\mathfrak{g}$ invariant subalgebra of $S(\mathfrak{g}^\vee)$.

For a given operation $(Z, \mathfrak{g}, j, l, s)$, one can construct the operation $(Z\hat{\otimes}W(\mathfrak{g}), \mathfrak{g}, j, l, s)$, where $\hat{\otimes}$ denotes graded tensor product. $(Z\hat{\otimes}W(\mathfrak{g}), \mathfrak{g}, j, l, s)$ is called equivariant extension of $(Z, \mathfrak{g}, j, l, s)$. The $\mathfrak{g}$ basic cohomology of $Z\hat{\otimes}W(\mathfrak{g})$ is called $\mathfrak{g}$ equivariant cohomology of Z. Representatives of $\mathfrak{g}$ equivariant cohomology classes of Z in $Z\hat{\otimes}W(\mathfrak{g})$ yield representatives of $\mathfrak{g}$ basic cohomology classes of Z by replacing the Weil generators ω, Ω, by a connection a, A of Z, i.e. a pair of elements a, A of $Z \otimes \mathfrak{g}$ of degree 1, 2, respectively, satisfying (2.2) with ω, Ω replaced by a, A (Weil homomorphism).

A homomorphism of the operation $(Z, \mathfrak{g}, j, l, s)$ into the operation $(Z', \mathfrak{g}', j', l', s')$ is a pair (f, μ), where $f : Z \to Z'$ is a degree zero graded algebra homomorphism, $\mu : \mathfrak{g}' \to \mathfrak{g}$ is a Lie algebra homomorphism and

$$j'(\xi') \circ f = f \circ j(\mu(\xi')), \qquad l'(\xi') \circ f = f \circ l(\mu(\xi')), \qquad s' \circ f = f \circ s \tag{2.3}$$

with $\xi' \in \mathfrak{g}'$. Clearly, (f, μ) induces a homomorphism of the $\mathfrak{g}$ basic cohomology of Z into the $\mathfrak{g}'$ basic cohomology of Z'.

If $\mu : \mathfrak{g}' \to \mathfrak{g}$ is a Lie algebra homomorphism, then the pair $(\mu^\vee, \mu)$ is a homomorphism of the Weil operation of $\mathfrak{g}$ $(W(\mathfrak{g}), \mathfrak{g}, j, l, s)$ into the Weil operation of $\mathfrak{g}'$ $(W(\mathfrak{g}'), \mathfrak{g}', j', l', s')$, where $\mu^\vee$ denotes the natural extension to $W(\mathfrak{g})$ of the dual linear homomorphism $\mu^\vee : \mathfrak{g}^\vee \to \mathfrak{g}'^\vee$. The generators ω, Ω, ω', Ω' of $W(\mathfrak{g})$, $W(\mathfrak{g}')$, respectively, obey the important relations

$$\mu^\vee(\omega) = \mu(\omega'), \qquad \mu^\vee(\Omega) = \mu(\Omega'). \tag{2.4}$$

By combining these identities with (2.3) with $f = \mu^\vee$, we find that the action of $j'(\xi')$, $l'(\xi')$. $\xi' \in \mathfrak{g}'$ and s' on $\mu(\omega')$, $\mu(\Omega')$ is obtained from that of $j(\mu(\xi'))$, $l(\mu(\xi'))$ and s on ω, Ω, Eq. (2.2), by applying $\mu^\vee$.

If (f, μ) is a homomorphism of the operations $(Z, \mathfrak{g}, j, l, s)$, $(Z', \mathfrak{g}', j', l', s')$, $(f \hat{\otimes} \mu^\vee, \mu)$ is a homomorphism of the corresponding equivariant extensions $(Z \hat{\otimes} W(\mathfrak{g}), \mathfrak{g}, j, l, s)$, $(Z' \hat{\otimes} W(\mathfrak{g}'), \mathfrak{g}', j', l', s')$ and induces a homomorphism of the $\mathfrak{g}$ equivariant cohomology of Z into the $\mathfrak{g}'$ equivariant cohomology of Z'.

If $(Z, \mathfrak{g}, j, l, s)$ is an operation and $\mu : \mathfrak{g}' \to \mathfrak{g}$ is a Lie algebra homomorphism, we can define a new operation $(Z, \mathfrak{g}', j', l', s')$ by setting

$$j'(\xi') = j(\mu(\xi')), \qquad l'(\xi') = l(\mu(\xi')), \qquad s' = s \tag{2.5}$$

with $\xi' \in \mathfrak{g}$. $(Z, \mathfrak{g}', j', l', s')$ is called pull-back of $(Z, \mathfrak{g}, j, l, s)$ by μ. In the particular case where $\mathfrak{g}'$ is a Lie subalgebra of $\mathfrak{g}$ and μ the natural inclusion, $(Z, \mathfrak{g}', j', l', s')$ is called restriction of $(Z, \mathfrak{g}, j, l, s)$ to $\mathfrak{g}'$. (id_Z, μ) is a homomorphism of the operations $(Z, \mathfrak{g}, j, l, s)$, $(Z, \mathfrak{g}', j', l', s')$, which induces an injection of the $\mathfrak{g}$ basic cohomology into the $\mathfrak{g}'$ basic cohomology of Z. Similarly, $(\mathrm{id}_Z \hat{\otimes} \mu^\vee, \mu)$ is a homomorphism of the associated equivariant extensions $(Z \hat{\otimes} W(\mathfrak{g}), \mathfrak{g}, j, l, s)$, $(Z \hat{\otimes} W(\mathfrak{g}'), \mathfrak{g}', j', l', s')$, which induces a homomorphism of the $\mathfrak{g}$ equivariant cohomology into the $\mathfrak{g}'$ equivariant cohomology of Z.

A wide class of operations is built as follows. Consider: (i) a Lie group $\mathcal{G}$ with Lie algebra $\mathrm{Lie}\,\mathcal{G}$; (ii) a principal $\mathcal{G}$ bundle $\pi_{\mathcal{P}} : \mathcal{P} \to \mathcal{M}$. Define $Z = \Omega^*(\mathcal{P})$, $\mathfrak{g} = \mathrm{Lie}\,\mathcal{G}$, $j(\xi) = j_{\mathcal{P}}(C_\xi)$, $l(\xi) = l_{\mathcal{P}}(C_\xi)$, for $\xi \in \mathfrak{g}$, and $s = d_{\mathcal{P}}$, where $\Omega^*(\mathcal{P})$ is the graded algebra of differential forms of $\mathcal{P}$, $j_{\mathcal{P}}$, $l_{\mathcal{P}}$ and $d_{\mathcal{P}}$ the customary differential geometric contraction, Lie derivative and de Rham differential operators, respectively, and C_ξ the vertical vector field corresponding to ξ. The resulting quintuplet $(Z, \mathfrak{g}, j, l, s)$ is an operation. It basic cohomology is isomorphic to the de Rham cohomology of the base $\mathcal{M}$.

Typically, in cohomological topological quantum field theory, $\mathcal{G}$ is a gauge group, $\mathcal{P}$ a supermanifold of gauge and matter fields propagating on a space–time manifold X and $\mathcal{M}$ some sort of gauge orbit or moduli space [11,12]. Representatives of equivariant classes, known as topological observables, play an important role, since they yield via the Weil homomorphism forms on $\mathcal{M}$, which can be used to probe its structure. They are obtained by integrating on cycles of X certain differential forms of X built with the fields. One is thus interested in mod d equivariant classes, where d is the Rham differential of X. This leads to the well-known descent formalism [18].

3. Generalities on Poisson manifolds

The target space of the Poisson sigma model is a Poisson manifold. We shall now briefly review the properties of Poisson manifolds relevant in the following discussion (see Ref. [17] for background material).

By definition, a manifold M is Poisson if it is equipped with a 2-vector ϖ^{ij} satisfying the relation

$$[\varpi, \varpi] = 0, \quad \text{i.e. } \varpi^{il}\partial_l \varpi^{jk} + \varpi^{jl}\partial_l \varpi^{ki} + \varpi^{kl}\partial_l \varpi^{ij} = 0. \tag{3.1}$$

Here and below, $[\cdot, \cdot]$ denotes the Schouten brackets, the only natural pairing of multivectors defined on any manifold.

On a Poisson manifold, one can define Poisson brackets of (local) functions f, g

$$\{f, g\} = \varpi^{ij}\partial_i f \partial_j g. \tag{3.2}$$

On account of (3.1), one indeed has

$$\{f, g\} + \{g, f\} = 0, \tag{3.3}$$

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0. \tag{3.4}$$

In this way, the space of functions $\mathrm{Fun}(M)$ becomes a Lie algebra, the Poisson algebra of M.

Functions f satisfying

$$[\varpi, f] = 0, \quad \text{i.e. } \varpi^{ij}\partial_j f = 0 \tag{3.5}$$

are called Casimir functions. Such f Poisson commute with any function g: $\{f, g\} = 0$. So, they form the center $\mathrm{Cas}(M)$ of $\mathrm{Fun}(M)$.

Vector fields u^i such that

$$[\varpi, u] = 0, \quad \text{i.e. } u^k\partial_k \varpi^{ij} - \partial_k u^i \varpi^{kj} - \partial_k u^j \varpi^{ik} = 0 \tag{3.6}$$

are called Poisson vector fields. Such u^i are precisely the vector fields leaving ϖ^{ij} invariant, since (3.6) states that $l_M(u)\varpi^{ij} = 0$, $l_M(u)$ being the usual Lie derivative. The Poisson vector fields u^i of the form

$$u_f = -[\varpi, f], \quad \text{i.e. } u_f^i = -\varpi^{ij}\partial_j f \tag{3.7}$$

for some function f are called Hamilton vector fields.

The Poisson vector fields span a Lie subalgebra $\mathrm{Pois}(M)$ of the Lie algebra $\mathrm{Vect}(M)$ of the vector fields of M. The Poisson vector fields are the natural symmetry Lie algebra of a Poisson manifold and of the associated geometrical structures. The Hamilton vector fields form a Lie subalgebra $\mathrm{Ham}(M)$ of $\mathrm{Pois}(M)$ as

$$[u_f, u_g] = u_{\{f,g\}} \tag{3.8}$$

for any two functions f, g. So, (3.7) establishes a canonical surjective Lie algebra homomorphism $\mathrm{Fun}(M) \to \mathrm{Ham}(M)$ with kernel $\mathrm{Cas}(M)$.

On a Poisson manifold, one can define a natural degree $+1$ operator acting on a general p-vector $\zeta^{i_1\cdots i_p}$:

$$q\zeta = [\varpi, \zeta]. \tag{3.9}$$

Using (3.1), it is easy to verify that

$$q^2 = 0. \tag{3.10}$$

Hence, a Poisson manifold has a natural notion of cohomology, called Poisson–Lichnerowicz cohomology. Since

$$[q, l_M(u)] = 0 \tag{3.11}$$

for any Poisson vector field u^i, one can define a Poisson (Hamilton) invariant Poisson–Lichnerowicz cohomology by restricting q to the complex of multivectors $\zeta^{i_1\cdots i_p}$ such that $l_M(u)\zeta^{i_1\cdots i_p} = 0$ for all Poisson vector fields u^i ($l_M(u_f)\zeta^{i_1\cdots i_p} = 0$ for all functions f).

The 0 Poisson–Lichnerowicz cocycles are the Casimir functions and span the zeroth Poisson–Lichnerowicz cohomology. The 1 Poisson–Lichnerowicz cocycles are the Poisson vector fields, the 1 Poisson–Lichnerowicz coboundaries are the Hamilton vector fields. So, the first Poisson–Lichnerowicz cohomology is the quotient of the Poisson by the Hamilton vector field spaces. It is possible to show that the second Poisson–Lichnerowicz cohomology describes the space of the infinitesimal deformation β^{ij} of the Poisson 2-vector ϖ^{ij} modulo the deformations of the form $l_M(u)\varpi^{ij}$ for some vector field u^i.

4. The Poisson and the Poisson equivariant operation of ΠTM

We are now going to construct the operations relevant in the following analysis. For a given Poisson manifold M, all these are $\mathfrak{H}$ operations over the graded associative algebra $\mathrm{Fun}(\Pi T\Pi T^*M)$ of functions of the superbundle $\Pi T\Pi T^*M$, or its equivariant extension $\mathrm{Fun}(\Pi T\Pi T^*M)\hat{\otimes}W(\mathfrak{H})$, where $\mathfrak{H}$ is some relevant Lie subalgebra of $\mathrm{Vect}(M)$ and Π the fiber parity inversion functor. Further, they all have a natural set of generators x^i, $X^{\sharp i}$, y_i, $Y_i^\sharp$ with the following degree assignments:

$$\deg x^i = 0, \qquad \deg X^{\sharp i} = 1, \qquad \deg y_i = 1, \qquad \deg Y_i^\sharp = 2 \tag{4.1}$$

and transformation properties under a change of local coordinates $t \to t'$:

$$\begin{aligned} &x'^i = t'^i(x), \qquad X^{\sharp' i} = \frac{\partial t'^i}{\partial t^j}(x)X^{\sharp j}, \qquad y'_i = \frac{\partial t^j}{\partial t'^i}(x)y_j, \\ &Y_i^{\sharp'} = \frac{\partial t^j}{\partial t'^i}(x)Y_j^\sharp + \left(\frac{\partial^2 t^j}{\partial t'^i \partial t'^k}\frac{\partial t'^k}{\partial t^l}\right)(x)X^{\sharp l}y_j. \end{aligned} \tag{4.2}$$

There are canonical injections of the spaces of p-vectors and p-forms of M into the graded algebra $\mathrm{Fun}(\Pi T\Pi T^*M)$ or its equivariant extension $\mathrm{Fun}(\Pi T\Pi T^*M)\hat{\otimes}W(\mathfrak{H})$. To any p-vector

$\beta^{i_1\cdots i_p}$, one associates

$$\beta(x,y)=\frac{1}{p!}\beta^{i_1\cdots i_p}(x)y_{i_1}\cdots y_{i_p}. \tag{4.3}$$

Similarly, to any p-form $\sigma_{i_1\cdots i_p}$, one associates

$$\sigma(x,X^\sharp)=\frac{1}{p!}\sigma_{i_1\cdots i_p}(x)X^{\sharp i_1}\cdots X^{\sharp i_p}. \tag{4.4}$$

These representations are very convenient and will be used throughout.

The fundamental differential operation of ΠTM is defined independently from any differential geometric structure on M. It is the Vect(M) operation over Fun($\Pi T\Pi T^*M$) with natural generators x^i, X^i, y_i, Y_i obeying

$$\begin{aligned}
&j(u)x^i=0,\qquad l(u)x^i=u^i(x),\qquad sx^i=X^i,\\
&j(u)X^i=u^i(x),\qquad l(u)X^i=\partial_j u^i(x)X^j,\qquad sX^i=0,\\
&j(u)y_i=0,\qquad l(u)y_i=-\partial_i u^j(x)y_j,\qquad sy_i=Y_i,\\
&j(u)Y_i=-\partial_i u^j(x)y_j,\qquad l(u)Y_i=-\partial_i u^j(x)Y_j-\partial_i\partial_j u^k(x)X^j y_k,\qquad sY_i=0
\end{aligned} \tag{4.5}$$

with u^i any vector field in Vect(M).

If M is a Poisson manifold with Poisson 2-vector ϖ^{ij}, the natural symmetry Lie algebra is the subalgebra of Vect(M) leaving ϖ^{ij} invariant, i.e. the Poisson Lie algebra Pois(M) (cf. Section 3). The Poisson differential operation of ΠTM is obtained by restricting the fundamental differential operation of ΠTM to Pois(M) (cf. Section 2). It is the Pois(M) operation over Fun($\Pi T\Pi T^*M$) with natural generators x^i, X^{*i}, y_i, Y_i^* obeying

$$\begin{aligned}
&j(u)x^i=0,\qquad l(u)x^i=u^i(x),\qquad sx^i=X^{*i}+\varpi^{ij}(x)y_j,\\
&j(u)X^{*i}=u^i(x),\quad l(u)X^{*i}=\partial_j u^i(x)X^{*j},\quad sX^{*i}=-\varpi^{ij}(x)Y_j^*-\partial_j\varpi^{ik}(x)X^{*j}y_k,\\
&j(u)y_i=0,\qquad l(u)y_i=-\partial_i u^j(x)y_j,\qquad sy_i=Y_i^*+\tfrac{1}{2}\partial_i\varpi^{jk}(x)y_jy_k,\\
&j(u)Y_i^*=-\partial_i u^j(x)y_j,\qquad l(u)Y_i^*=-\partial_i u^j(x)Y_j^*-\partial_i\partial_j u^k(x)X^{*j}y_k,\\
&sY_i^*=-\tfrac{1}{2}\partial_i\partial_j\varpi^{kl}(x)X^{*j}y_ky_l+\partial_i\varpi^{jk}(x)y_jY_k^*,
\end{aligned} \tag{4.6}$$

where now u^i is any Poisson vector field in Pois(M) (cf. Eq. (3.6)). The Poisson generators X^{*i}, Y_i^* are related to the fundamental generators X^i, Y_i by the simple shifts

$$X^{*i}=X^i-\varpi^{ij}(x)y_j,\qquad Y_i^*=Y_i-\tfrac{1}{2}\partial_i\varpi^{jk}(x)y_jy_k. \tag{4.7}$$

Such shifts have a simple geometrical interpretation:

$$qx^i=-\varpi^{ij}(x)y_j,\qquad qy_i=-\tfrac{1}{2}\partial_i\varpi^{jk}(x)y_jy_k, \tag{4.8}$$

where q is defined in (3.9).

Next, we consider the Weil operation of the Lie algebra Pois(M) (cf. Section 2). It is the Pois(M) operation over the Weil algebra W(Pois(M)) with natural Poisson vector field valued generators ω^i, Ω^i satisfying the form of the Weil operation relations (2.2) appropriate for $\mathrm{Pois}(M)$. It turns out to be more efficient to combine ω^i, Ω^i with the generators x^i to form composites $\omega^i(x)$, $\Omega^i(x)$, by replacing the dummy local coordinates t^i appearing in the local expression $\omega^i(t)$, $\Omega^i(t)$ of ω^i, Ω^i by x^i. The Poisson–Weil generators $\omega^i(x)$, $\Omega^i(x)$ carry degrees:

$$\deg \omega^i(x) = 1, \qquad \deg \Omega^i(x) = 2 \tag{4.9}$$

by being Poisson vector fields satisfy

$$\begin{aligned}
&(\omega^k \partial_k \varpi^{ij} - \partial_k \omega^i \varpi^{kj} - \partial_k \omega^j \varpi^{ik})(x) = 0, \\
&(\Omega^k \partial_k \varpi^{ij} - \partial_k \Omega^i \varpi^{kj} - \partial_k \Omega^j \varpi^{ik})(x) = 0
\end{aligned} \tag{4.10}$$

and obey

$$\begin{aligned}
&j(u)\omega^i(x) = u^i(x), \qquad l(u)\omega^i(x) = \omega^j \partial_j u^i(x), \\
&s\omega^i(x) = \Omega^i(x) - \omega^j \partial_j \omega^i(x) + (X^{*j} + \varpi^{jk}(x) y_k)\partial_j \omega^i(x), \\
&j(u)\Omega^i(x) = 0, \qquad l(u)\Omega^i(x) = \Omega^j \partial_j u^i(x), \\
&s\Omega^i(x) = -\omega^j \partial_j \Omega^i(x) + \Omega^j \partial_j \omega^i(x) + (X^{*j} + \varpi^{jk}(x) y_k)\partial_j \Omega^i(x),
\end{aligned} \tag{4.11}$$

where u^i is any Poisson vector field in $\mathrm{Pois}(M)$.

The equivariant extension of the Poisson differential operation of ΠTM (cf. Section 1), referred to as Poisson equivariant differential operation in the following, is now easily worked out. It is the $\mathrm{Pois}(M)$ operation over $\mathrm{Fun}(\Pi T \Pi T^* M) \hat{\otimes} W(\mathrm{Pois}(M))$ with natural generators x^i, $\tilde{X}^i$, y_i, $\tilde{Y}_i$, $\omega^i(x)$, $\Omega^i(x)$ satisfying

$$\begin{aligned}
&j(u)x^i = 0, \qquad l(u)x^i = u^i(x), \qquad sx^i = \tilde{X}^i + \varpi^{ij}(x) y_j + \omega^i(x), \\
&j(u)\tilde{X}^i = 0, \qquad l(u)\tilde{X}^i = \partial_j u^i(x) \tilde{X}^j, \\
&s\tilde{X}^i = -\varpi^{ij}(x)\tilde{Y}_j - \partial_j \varpi^{ik}(x)\tilde{X}^j y_k - \Omega^i(x) + \partial_j \omega^i(x)\tilde{X}^j, \\
&j(u)y_i = 0, \qquad l(u)y_i = -\partial_i u^j(x) y_j, \\
&sy_i = \tilde{Y}_i + \tfrac{1}{2}\partial_i \varpi^{jk}(x) y_j y_k - \partial_i \omega^j(x) y_j, \\
&j(u)\tilde{Y}_i = 0, \qquad l(u)\tilde{Y}_i = -\partial_i u^j(x)\tilde{Y}_j - \partial_i \partial_j u^k(x)\tilde{X}^j y_k, \\
&s\tilde{Y}_i = -\tfrac{1}{2}\partial_i \partial_j \varpi^{kl}(x)\tilde{X}^j y_k y_l + \partial_i \varpi^{jk}(x) y_j \tilde{Y}_k \\
&\qquad + \partial_i \Omega^j(x) y_j - \partial_i \omega^j(x)\tilde{Y}_j - \partial_i \partial_j \omega^k(x)\tilde{X}^j y_k,
\end{aligned} \tag{4.12}$$

$$\begin{aligned}
&j(u)\omega^i(x) = u^i(x), \qquad l(u)\omega^i(x) = \omega^j \partial_j u^i(x), \\
&s\omega^i(x) = \Omega^i(x) + (\tilde{X}^j + \varpi^{jk}(x) y_k)\partial_j \omega^i(x), \\
&j(u)\Omega^i(x) = 0, \qquad l(u)\Omega^i(x) = \Omega^j \partial_j u^i(x), \\
&s\Omega^i(x) = \Omega^j \partial_j \omega^i(x) + (\tilde{X}^j + \varpi^{jk}(x) y_k)\partial_j \Omega^i(x),
\end{aligned} \tag{4.13}$$

where again u^i is any Poisson vector field in $\mathrm{Pois}(M)$. The Poisson equivariant generators $\tilde{X}^i$, $\tilde{Y}_i$ are related to the Poisson generators X^{*i}, Y^*_i by the simple shifts

$$\tilde{X}^i = X^{*i} - \omega^i(x), \qquad \tilde{Y}_i = Y^*_i + \partial_i \omega^j(x) y_j. \tag{4.14}$$

The shifts have a simple formal geometrical interpretation:

$$l(\omega)x^i = \omega^i(x), \qquad l(\omega)y_i = -\partial_i\omega^j(x)y_j. \tag{4.15}$$

Note that all the Poisson equivariant generators but $\omega^i(x)$ are horizontal, i.e. they are annihilated by every $j(u)$ for all Poisson vector fields u^i in $\mathrm{Pois}(M)$.

5. The Hamilton and the Hamilton equivariant operation of ΠTM

As discussed in Section 3, if M is a Poisson manifold, there is a canonical Lie algebra homomorphism $\varrho : \mathrm{Fun}(M) \to \mathrm{Pois}(M)$, defined by (3.7), of the Poisson algebra $\mathrm{Fun}(M)$ into the Poisson vector field Lie algebra $\mathrm{Pois}(M)$, whose image is the Hamilton vector field Lie subalgebra $\mathrm{Ham}(M)$.

The Hamilton differential operation of ΠTM is the pull-back of the Poisson differential operation of ΠTM by the Lie algebra homomorphism ϱ (cf. Section 2). It is the $\mathrm{Fun}(M)$ operation over $\mathrm{Fun}(\Pi T\Pi T^*M)$ with natural generators x^i, X^{*i}, y_i, Y^*_i satisfying (4.6) with $j(u)$, $l(u)$, u^i substituted by $j(f)$, $l(f)$, $u^i{}_f$ (cf. Eq. (3.7)), respectively, for any function f of $\mathrm{Fun}(M)$. The Hamilton generators X^{*i}, Y^*_i are still related to the fundamental generators X^i, Y_i by (4.7).

Next, we consider the Weil operation of the Lie algebra $\mathrm{Fun}(M)$ (cf. Section 2). It is the $\mathrm{Fun}(M)$ operation over the Weil algebra $W(\mathrm{Fun}(M))$ with natural scalar valued generators ϕ, Φ satisfying the form of the Weil operation relations (2.2) appropriate for $\mathrm{Fun}(M)$. As in the Poisson case, it turns out to be more efficient to combine ϕ, Φ with the generators x^i to form composites $\phi(x)$, $\Phi(x)$. The Hamilton–Weil generators $\phi(x)$, $\Phi(x)$ carry degrees

$$\deg\phi(x) = 1, \qquad \deg\Phi(x) = 2 \tag{5.1}$$

and satisfy

$$\begin{aligned}
&j(f)\phi(x) = f(x), \qquad l(f)\phi(x) = 0,\\
&s\phi(x) = \Phi(x) - \tfrac{1}{2}\varpi^{ij}(x)\partial_i\phi(x)\partial_j\phi(x) + (X^{*i} + \varpi^{ij}(x)y_j)\partial_i\phi(x),\\
&j(f)\Phi(x) = 0, \qquad l(f)\Phi(x) = 0,\\
&s\Phi(x) = -\varpi^{ij}(x)\partial_i\phi(x)\partial_j\Phi(x) + (X^{*i} + \varpi^{ij}(x)y_j)\partial_i\Phi(x)
\end{aligned} \tag{5.2}$$

for any function f in $\mathrm{Fun}(M)$.

From the discussion of Section 2, we know that the image by ϱ of the Hamilton–Weil generators $\phi(x)$, $\Phi(x)$ equals the image by $\varrho^\vee$ of the Poisson–Weil generators $\omega^i(x)$, $\Omega^i(x)$, respectively. Explicitly, denoting these objects by $\omega^i_\phi(x)$, $\Omega^i_\Phi(x)$, one has

$$\omega^i_\phi(x) = -\varpi^{ij}(x)\partial_j\phi(x), \qquad \Omega^i_\Phi(x) = -\varpi^{ij}(x)\partial_j\Phi(x). \tag{5.3}$$

Using (5.1) and (5.2), it is easy to check that $\omega^i_\phi(x)$, $\Omega^i_\Phi(x)$ fulfill (4.9) and (4.10) and satisfy (4.11) with $j(u)$, $l(u)$, u^i, $\omega^i(x)$, $\Omega^i(x)$ substituted by $j(f)$, $l(f)$, $u^i{}_f$, $\omega^i_\phi(x)$, $\Omega^i_\Phi(x)$, respectively, for any function f of $\mathrm{Fun}(M)$.

The equivariant extension of the Hamilton differential operation of ΠTM (cf. Section 2), referred to as Hamilton equivariant differential operation below, is now easily obtained. It is the $\mathrm{Fun}(M)$ operation over $\mathrm{Fun}(\Pi T\Pi T^*M)\hat{\otimes}W(\mathrm{Fun}(M))$ with natural generators x^i, $\tilde{X}^i$, y_i, $\tilde{Y}_i$, $\phi(x)$, $\Phi(x)$ satisfying (4.12) with $j(u)$, $l(u)$, u^i, $\omega^i(x)$, $\Omega^i(x)$ substituted by $j(f)$, $l(f)$, $u^i{}_f$, $\omega^i_\phi(x)$, $\Omega^i_\Phi(x)$, respectively, and

$$\begin{aligned}
&j(f)\phi(x) = f(x), \qquad l(f)\phi(x) = 0,\\
&s\phi(x) = \Phi(x) + \tfrac{1}{2}\varpi^{ij}(x)\partial_i\phi(x)\partial_j\phi(x) + (\tilde{X}^i + \varpi^{ij}(x)y_j)\partial_i\phi(x),\\
&j(f)\Phi(x) = 0, \qquad l(f)\Phi(x) = 0, \qquad s\Phi(x) = (\tilde{X}^i + \varpi^{ij}(x)y_j)\partial_i\Phi(x)
\end{aligned} \tag{5.4}$$

for any function f of $\mathrm{Fun}(M)$. The Hamilton equivariant generators $\tilde{X}^i$, $\tilde{Y}_i$ are related the Hamilton generators X^{*i}, Y^*_i by (4.14) with $\omega^i(x)$, $\Omega^i(x)$ substituted by $\omega^i_\phi(x)$, $\Omega^i_\Phi(x)$, respectively. Obviously, $\omega^i_\phi(x)$, $\Omega^i_\Phi(x)$ satisfy (4.13) with $j(u)$, $l(u)$, u^i, $\omega^i(x)$, $\Omega^i(x)$ substituted by $j(f)$, $l(f)$, $u^i{}_f$, $\omega^i_\phi(x)$, $\Omega^i_\Phi(x)$, respectively, for any function f of $\mathrm{Fun}(M)$, as usual.

As the above construction is completely local, it works also for the local Hamilton symmetry at the price of having multivalued Hamilton–Weil generators $\phi(x)$, $\Phi(x)$. This may be relevant in the analysis of the implications of the global topology of M.

6. The differential d

There is an important operator d which enters the construction of the topological observables of the Poisson sigma model. For the sake of clarity, we shall analyze its properties separately in this section.

d is the degree +1 derivation on $\mathrm{Fun}(\Pi T\Pi T^*M)\hat{\otimes}W(\mathrm{Pois}(M))$ defined in terms of the Poisson equivariant generators by

$$\mathrm{d}x^i = \tilde{X}^i, \qquad \mathrm{d}\tilde{X}^i = 0, \qquad \mathrm{d}y_i = \tilde{Y}_i, \qquad \mathrm{d}\tilde{Y}_i = 0, \tag{6.1}$$

$$\mathrm{d}\omega^i(x) = \Omega^i(x) + \tilde{X}^j\partial_j\omega^i(x), \qquad \mathrm{d}\Omega^i(x) = \tilde{X}^j\partial_j\Omega^i(x). \tag{6.2}$$

The interest of d stems from the fact that it is nilpotent and (anti)commutes with all the derivations of the Poisson equivariant operation of ΠTM:

$$[\mathrm{d},\mathrm{d}] = 0, \qquad [\mathrm{d}, j(u)] = 0, \qquad [\mathrm{d}, l(u)] = 0, \qquad [\mathrm{d}, s] = 0 \tag{6.3}$$

for any Poisson vector u^i field in $\mathrm{Pois}(M)$.

d can be defined also in $\mathrm{Fun}(\Pi T\Pi T^*M)\hat{\otimes}W(\mathrm{Fun}(M))$ in terms of the Hamilton equivariant generators by the same relations (6.1) and by

$$\mathrm{d}\phi(x) = \Phi(x) + \tilde{X}^i\partial_i\phi(x), \qquad \mathrm{d}\Phi(x) = \tilde{X}^i\partial_i\Phi(x). \tag{6.4}$$

(6.3) holds also in this case but with $j(u)$, $l(u)$ replaced by $j(f)$, $l(f)$, for any function f in $\mathrm{Fun}(M)$.

It is easy to check, using (6.4), that $\omega^i_\phi(x)$, $\Omega^i_\Phi(x)$, given by (5.3), satisfy relations (6.2).

7. Mod d Poisson and Hamilton equivariant classes of ΠTM

To construct topological observables of the Poisson sigma model, one needs representatives $\mathcal{O}$ of mod d Poisson equivariant classes. By definition, any such $\mathcal{O}$ is an element of $\mathrm{Fun}(\Pi T\Pi T^*M)\hat{\otimes}W(\mathrm{Pois}(M))$ satisfying

$$j(u)\mathcal{O} = \mathrm{d}\mathcal{O}_{-1}(u), \qquad l(u)\mathcal{O} = \mathrm{d}\mathcal{O}_0(u), \qquad s\mathcal{O} = \mathrm{d}\mathcal{O}_{+1} \tag{7.1}$$

for some $\mathcal{O}_{-1}(u)$, $\mathcal{O}_0(u)$, $\mathcal{O}_{+1}$ in $\mathrm{Fun}(\Pi T\Pi T^*M)\hat{\otimes}W(\mathrm{Pois}(M))$, for any Poisson vector u^i field in $\mathrm{Pois}(M)$. An analogous definition holds when restricting to the Hamilton symmetry with $\mathcal{O}$, $\mathcal{O}_{-1}(u)$, $\mathcal{O}_0(u)$, $\mathcal{O}_{+1}$ substituted by elements $\mathcal{O}$, $\mathcal{O}_{-1}(f)$, $\mathcal{O}_0(f)$, $\mathcal{O}_{+1}$ of $\mathrm{Fun}(\Pi T\Pi T^*M)\hat{\otimes}W(\mathrm{Fun}(M))$, for any function f in $\mathrm{Fun}(M)$.

Let j_M, l_M and d_M denote the usual differential geometric contraction, Lie derivative and de Rham differential operators of M.

Let $\beta^{i_1\cdots i_p}$ be any p-vector, which we represent in $\mathrm{Fun}(\Pi T\Pi T^*M)\hat{\otimes}W(\mathrm{Pois}(M))$ as usual as

$$\beta(x, y) = \frac{1}{p!}\beta^{i_1\cdots i_p}(x)y_{i_1}\cdots y_{i_p}. \tag{7.2}$$

Using (4.12), by a simple calculation, one finds

$$\begin{aligned}
&j(u)\beta(x, y) = 0, \qquad l(u)\beta(x, y) = l_M(u)\beta(x, y),\\
&s\beta(x, y) = \mathrm{d}\beta(x, y) - [\varpi, \beta](x, y) + l_M(\omega)\beta(x, y)
\end{aligned} \tag{7.3}$$

for any Poisson vector field u^i in $\mathrm{Pois}(M)$. If $\beta(x, y)$ is a representative of a Poisson invariant Poisson–Lichnerowicz cohomology class, i.e.:

$$l_M(u)\beta(x, y) = 0, \qquad q\beta(x, y) = 0 \tag{7.4}$$

for any Poisson vector field u^i in $\mathrm{Pois}(M)$ (cf. Section 3, Eq. (3.9)), then

$$j(u)\beta(x, y) = 0, \qquad l(u)\beta(x, y) = 0, \qquad s\beta(x, y) = \mathrm{d}\beta(x, y) \tag{7.5}$$

for all with u^i in $\mathrm{Pois}(M)$. Thus, $\beta(x, y)$ is a representative of a mod d Poisson equivariant cohomology class.

Let $\sigma_{i_1\cdots i_p}$ be any p-form, which we represent in $\mathrm{Fun}(\Pi T\Pi T^*M)\hat{\otimes}W(\mathrm{Pois}(M))$ as

$$\sigma(x, \tilde{X}) = \frac{1}{p!}\sigma_{i_1\cdots i_p}(x)\tilde{X}^{i_1}\cdots\tilde{X}^{i_p}. \tag{7.6}$$

Using (4.12) again, one finds

$$\begin{aligned}
&j(u)\sigma(x, \tilde{X}) = 0, \qquad l(u)\sigma(x, \tilde{X}) = l_M(u)\sigma(x, \tilde{X}),\\
&s\sigma(x, \tilde{X}) = d_M\sigma(x, \tilde{X}) + l_M(\omega)\sigma(x, \tilde{X})\\
&\qquad\qquad - j_M(\Omega)\sigma(x, \tilde{X}), + kd_M\sigma(x, \tilde{X}) - \mathrm{d}k\,\sigma(x, \tilde{X})
\end{aligned} \tag{7.7}$$

for every Poisson vector field u^i in $\mathrm{Pois}(M)$, where the operator k is the degree 0 derivation defined by

$$k\tilde{X}^i = \varpi^{ij}(x)y_j \tag{7.8}$$

and acting trivially on all other Poisson equivariant generators. Therefore, if $\sigma(x, \tilde{X})$ is a representative of a Poisson basic de Rham cohomology class, i.e.:

$$j_M(u)\sigma(x, \tilde{X}) = 0, \qquad l_M(u)\sigma(x, \tilde{X}) = 0, \qquad d_M\sigma(x, \tilde{X}) = 0 \tag{7.9}$$

for any Poisson vector field u^i in $\mathrm{Pois}(M)$, then

$$j(u)\sigma(x, \tilde{X}) = 0, \qquad l(u)\sigma(x, \tilde{X}) = 0, \qquad s\sigma(x, \tilde{X}) = -\mathrm{d}k\,\sigma(x, \tilde{X}) \tag{7.10}$$

for all with u^i in $\mathrm{Pois}(M)$. Thus, $\sigma(x, \tilde{X})$ is a representative of a mod d Poisson equivariant cohomology class.

Demanding invariance or basicity under the Poisson symmetry is very restrictive and in general only trivial or uninteresting solutions of this requirement are available on a generic Poisson manifold. So, it is important to see whether restricting to the Hamilton symmetry yields mod d Hamilton equivariant cohomology classes other than those obtained from the mod d Poisson equivariant ones via pull-back by the homomorphism $\varrho : \mathrm{Fun}(M) \to \mathrm{Pois}(M)$ (cf. Section 4).

Consider again a p-vector $\beta^{i_1\cdots i_p}$ and view $\beta(x, y)$ as an element of $\mathrm{Fun}(\Pi T\Pi T^*M)\hat{\otimes} W(\mathrm{Fun}(M))$. Proceeding as in (7.3), one finds

$$\begin{aligned} &j(f)\beta(x, y) = 0, \qquad l(f)\beta(x, y) = [[\varpi, \beta], f](x, y) - [\varpi, [\beta, f]](x, y),\\ &s\beta(x, y) = \mathrm{d}\beta(x, y) - [\varpi, \beta](x, y) + [[\varpi, \beta], \phi](x, y) - [\varpi, [\beta, \phi]](x, y) \end{aligned} \tag{7.11}$$

for any function f in $\mathrm{Fun}(M)$. If $\beta(x, y)$ satisfies

$$[f, \beta](x, y) = 0, \qquad q\beta(x, y) = 0 \tag{7.12}$$

for any function f in $\mathrm{Fun}(M)$, and is therefore a representative of a Hamilton invariant Poisson–Lichnerowicz cohomology class (cf. Section 3, Eq. (3.9)), then

$$j(f)\beta(x, y) = 0, \qquad l(f)\beta(x, y) = 0, \qquad s\beta(x, y) = \mathrm{d}\beta(x, y) \tag{7.13}$$

for all f in $\mathrm{Fun}(M)$. Thus, $\beta(x, y)$ is a representative of a mod d Hamilton equivariant cohomology class.

Consider again a p-form $\sigma_{i_1\cdots i_p}$ and view $\sigma(x, \tilde{X})$ as an element of $\mathrm{Fun}(\Pi T\Pi T^*M)\hat{\otimes} W(\mathrm{Fun}(M))$. Proceeding as in (7.7) and performing some simple rearrangements, one finds

$$\begin{aligned} &j(f)\sigma(x, \tilde{X}) = 0, \qquad l(f)\sigma(x, \tilde{X}) = \mathrm{d}\,j_M(u_f)\sigma(x, \tilde{X}) + j_M(u_f)d_M\sigma(x, \tilde{X}),\\ &s\sigma(x, \tilde{X}) = \mathrm{d}(\sigma(x, \tilde{X}) - h\sigma(x, \tilde{X})) + hd_M\sigma(x, \tilde{X}) \end{aligned} \tag{7.14}$$

for every function f in $\mathrm{Fun}(M)$, where the operator h is the degree 0 derivation defined by

$$h\tilde{X}^i = \varpi^{ij}(x)(y_j - \partial_j\phi(x)) \tag{7.15}$$

and acting trivially on all other Hamilton equivariant generators and u^i_f is defined in (3.7). Therefore, if σ satisfies the condition

$$kd_M\sigma(x, \tilde{X}) = 0, \tag{7.16}$$

then

$$j(f)\sigma(x,\tilde{X})=0,\qquad l(f)\sigma(x,\tilde{X})=\mathrm{d}j_M(u_f)\sigma(x,\tilde{X}),$$
$$s\sigma(x,\tilde{X})=\mathrm{d}(\sigma(x,\tilde{X})-h\sigma(x,\tilde{X}))\tag{7.17}$$

for all f in $\mathrm{Fun}(M)$. Thus, $\sigma(x,\tilde{X})$ is a representative of a mod d Hamilton equivariant cohomology class.

8. Poisson and Hamilton action of a Lie algebra $\mathfrak{h}$

In the analysis of the Poisson sigma model expounded in later sections, it turns out to be natural to restrict the symmetry Lie algebra to be some finite-dimensional Lie subalgebra of the Poisson or Hamilton vector field Lie algebras. This can be done efficiently by using the formalism of Poisson or Hamilton actions on M of some abstract finite-dimensional Lie algebra $\mathfrak{h}$.

Let $\mathfrak{h}$ be a Lie algebra and let $\{t_a\}$ be a basis of $\mathfrak{h}$. Then:

$$[t_a,t_b]=c_{ab}^c t_c,\tag{8.1}$$

c_{ab}^c being the structure constants of $\mathfrak{h}$.

A Poisson (Hamilton) action of $\mathfrak{h}$ on M is a Lie algebra homomorphism $\upsilon:\mathfrak{h}\to\mathrm{Pois}(M)$ ($\varsigma:\mathfrak{h}\to\mathrm{Fun}(M)$). In the Poisson case, $\upsilon(\mathfrak{h})$ is a Lie subalgebra of $\mathrm{Pois}(M)$. Indeed, setting $\upsilon_a^i=\upsilon^i(t_a)$, one has

$$[\upsilon_a,\upsilon_b]=c_{ab}^c\upsilon_c.\tag{8.2}$$

Similarly, in the Hamilton case, $\varsigma(\mathfrak{h})$ is a Lie subalgebra of $\mathrm{Fun}(M)$. Setting $h_a=\varsigma(t_a)$, one has

$$\{h_a,h_b\}=c_{ab}^c h_c.\tag{8.3}$$

The $\mathfrak{h}$ Poisson (Hamilton) differential operation of ΠTM is the pull-back of the Poisson (Hamilton) differential operation of ΠTM by the Lie algebra homomorphism υ (ς) (cf. Sections 2, 4 and 5). Hence, it is the $\mathfrak{h}$ operation over $\mathrm{Fun}(\Pi T\Pi T^*M)$ with natural generators x^i, X^{*i}, y_i, Y_i^* satisfying (4.6) with $j(u)$, $l(u)$, u^i ($j(f)$, $l(f)$, f) substituted by $j(r)$, $l(r)$, $\upsilon^i(r)$ ($u^i_{\varsigma(r)}$), respectively, for any element r of $\mathfrak{h}$. The $\mathfrak{h}$ Poisson (Hamilton) generators X^{*i}, Y_i^* are related to the fundamental generators X^i, Y_i again by (4.7).

Next, we consider the Weil operation of the Lie algebra $\mathfrak{h}$ (cf. Section 2). It is the $\mathfrak{h}$ operation over the Weil algebra $W(\mathfrak{h})$ with natural generators γ^a, Γ^a dual to the basis vector t_a of degrees

$$\deg\gamma^a=1,\qquad\deg\Gamma^a=2\tag{8.4}$$

and satisfying

$$j(r)\gamma^a=r^a,\qquad l(r)\gamma^a=-c_{bc}^a r^b\gamma^c,\qquad s\gamma^a=\Gamma^a-\tfrac{1}{2}c_{bc}^a\gamma^b\gamma^c,$$
$$j(r)\Gamma^a=0,\qquad l(r)\Gamma^a=-c_{bc}^a r^b\Gamma^c,\qquad s\Gamma^a=-c_{bc}^a\gamma^b\Gamma^c\tag{8.5}$$

with r in $\mathfrak{h}$.

From the discussion of Section 2, we know that the image by $\upsilon(\varsigma)$ of the $\mathfrak{h}$ Weil generators γ^a, Γ^a equals the image by $\upsilon^\vee$ $(\varsigma^\vee)$ of the Poisson–Weil (Hamilton–Weil) generators $\omega^i(x)$, $\Omega^i(x)$, $(\phi(x), \Phi(x))$, respectively. Explicitly, denoting these objects by $\omega^i_\gamma(x)$, $\Omega^i_\Gamma(x)$, $(\phi_\gamma(x), \Phi_\Gamma(x))$, one has

$$\omega^i_\gamma(x) = \sum_a \gamma^a v^i_a(x), \qquad \Omega^i_\Gamma(x) = \sum_a \Gamma^a v^i_a(x), \tag{8.6}$$

$$\left(\phi_\gamma(x) = \sum_a \gamma^a h_a(x), \qquad \Phi_\Gamma(x) = \sum_a \Gamma^a h_a(x)\right). \tag{8.7}$$

Using (8.4) and (8.5), it is easy to check that $\omega^i_\gamma(x)$, $\Omega^i_\Gamma(x)(\phi_\gamma(x), \Phi_\Gamma(x))$ fulfill (4.9) and (4.10) ((5.1)) and satisfy (4.11) ((5.2)) with $j(u)$, $l(u)$, u^i, $\omega^i(x)$, $\Omega^i(x)(j(f), l(f), f, \phi(x), \Phi(x))$ substituted by $j(r)$, $l(r)$, $\upsilon^i(r)$, $\omega^i_\gamma(x)$, $\Omega^i_\Gamma(x)(u^i_{\varsigma(r)}, \phi_\gamma(x), \Phi_\Gamma(x))$, respectively, for r in $\mathfrak{h}$.

The equivariant extension of the $\mathfrak{h}$ Poisson (Hamilton) differential operation of ΠTM (cf. Section 2), which we shall call $\mathfrak{h}$ Poisson (Hamilton) equivariant differential operation below, is now easily obtained. It is the $\mathfrak{h}$ operation over $\mathrm{Fun}(\Pi T\Pi T^*M)\hat{\otimes}W(\mathfrak{h})$ with natural generators x^i, $\tilde{X}^i$, y_i, $\tilde{Y}_i$, γ^a, Γ^a satisfying (4.12), with $j(u)$, $l(u)$, u^i, $\omega^i(x)$, $\Omega^i(x)$ $(j(f), l(f), f, \omega^i_\phi(x), \Omega^i_\Phi(x))$ substituted by $j(r)$, $l(r)$, $\upsilon^i(r)$, $\omega^i_\gamma(x)$, $\Omega^i_\Gamma(x)(u^i_{\varsigma(r)}$, $\omega^i_{\phi_\gamma}(x)$, $\Omega^i_{\Phi_\Gamma}(x))$, respectively, and (8.5), for any element r of $\mathfrak{h}$. The $\mathfrak{h}$ Poisson (Hamilton) equivariant generators $\tilde{X}^i$, $\tilde{Y}_i$ are related the $\mathfrak{h}$ Poisson (Hamilton) generators X^{*i}, Y^*_i by (4.14) with $\omega^i(x)$, $\Omega^i(x)(\omega^i_\phi(x), \Omega^i_\Phi(x))$ substituted by $\omega^i_\gamma(x)$, $\Omega^i_\Gamma(x)(\omega^i_{\phi_\gamma}(x), \Omega^i_{\Phi_\Gamma}(x))$ (cf. Eq. (5.3)), respectively.

The d operator is defined in obvious fashion:

$$\mathrm{d}\gamma^a = \Gamma^a, \qquad \mathrm{d}\Gamma^a = 0. \tag{8.8}$$

(6.3) holds with $j(u)$, $l(u)$ substituted by $j(r)$, $l(r)$, respectively, for r in $\mathfrak{h}$.

Representatives of mod d $\mathfrak{h}$ Poisson (Hamilton) equivariant classes are obtained from those of mod d Poisson and Hamilton equivariant classes discussed in Section 7 by pull-back via the Poisson (Hamilton) action $\upsilon(\varsigma)$ of $\mathfrak{h}$ on M.

9. Two-dimensional de Rham superfields and singular superchains

In general, the fields of a two-dimensional field theory are differential forms on a two-dimensional manifold Σ. They can be viewed as elements of the space $\mathrm{Fun}(\Pi T\Sigma)$ of functions on the parity reversed tangent bundle $\Pi T\Sigma$ of Σ, which we shall call de Rham superfields [6]. More explicitly, we associate to the coordinates z^α of Σ Grassmann odd partners ζ^α with

$$\deg z^\alpha = 0, \qquad \deg \zeta^\alpha = 1 \tag{9.1}$$

and

$$\mathrm{d}z^\alpha = \zeta^\alpha, \qquad \mathrm{d}\zeta^\alpha = 0. \tag{9.2}$$

A generic de Rham superfield $\Psi(z,\zeta)$ is a triplet formed by a 0-, 1- and 2-form field $\psi^{(0)}(z)$, $\psi^{(1)}_\alpha(z)$ and $\psi^{(2)}_{\alpha\beta}(z)$ organized as

$$\Psi(z,\zeta) = \psi^{(0)}(z) + \zeta^\alpha \psi^{(1)}_\alpha(z) + \tfrac{1}{2}\zeta^\alpha\zeta^\beta\psi^{(2)}_{\alpha\beta}(z). \tag{9.3}$$

Note that in this formalism, the de Rham differential d of Σ is simply

$$\mathrm{d} = \zeta^\alpha \frac{\partial}{\partial z^\alpha}. \tag{9.4}$$

The coordinate invariant integration measure is

$$\mu = \mathrm{d}z^1\,\mathrm{d}z^2\,\mathrm{d}\zeta^1\,\mathrm{d}\zeta^2. \tag{9.5}$$

Any de Rham superfield Ψ can be integrated on Σ according to the prescription:

$$\int_\Sigma \mu\Psi = \int_\Sigma \frac{1}{2}\,\mathrm{d}z^\alpha\,\mathrm{d}z^\beta\psi^{(2)}_{\alpha\beta}(z). \tag{9.6}$$

By Stokes' theorem:

$$\int_\Sigma \mu\,\mathrm{d}\Psi = 0. \tag{9.7}$$

The singular chain complex of Σ can be given a parallel treatment. A singular superchain C is a triplet formed by a zero-, one- and two-dimensional singular chain $C_{(0)}$, $C_{(1)}$, $C_{(2)}$ organized as a formal sum:

$$C = C_{(0)} + C_{(1)} + C_{(2)}. \tag{9.8}$$

The singular boundary operator ∂ extends to superchains in obvious fashion by setting

$$(\partial C)_{(0)} = \partial C_{(1)}, \qquad (\partial C)_{(1)} = \partial C_{(2)}, \qquad (\partial C)_{(2)} = 0. \tag{9.9}$$

A singular supercycle Z is a superchain such that

$$\partial Z = 0. \tag{9.10}$$

A de Rham superfield Ψ can be integrated on a superchain C:

$$\int_C \mu\Psi = \int_{C_{(0)}} \psi^{(0)} + \int_{C_{(1)}} \mathrm{d}z^\alpha\,\psi^{(1)}_\alpha(z) + \int_{C_{(2)}} \frac{1}{2}\,\mathrm{d}z^\alpha\,\mathrm{d}z^\beta\psi^{(2)}_{\alpha\beta}(z). \tag{9.11}$$

Stokes' theorem states that

$$\int_C \mu\,\mathrm{d}\Psi = \int_{\partial C} \mu\Psi. \tag{9.12}$$

In particular, if Z is a supercycle:

$$\int_Z \mu\,\mathrm{d}\Psi = 0. \tag{9.13}$$

In the case where Σ has a non-empty boundary $\partial\Sigma$, the above relations hold provided the component fields of the superfield obey suitable boundary conditions [6].

10. The $\mathfrak{h}$ Poisson sigma model

The Poisson Sigma Model is a two-dimensional field theory whose base space is a closed two-dimensional surface Σ and whose target space is a Poisson manifold M.

We assume that a finite-dimensional Lie algebra $\mathfrak{h}$ is given together with a Hamilton action of $\mathfrak{h}$ on M $\varsigma : \mathfrak{h} \to \mathrm{Fun}(M)$ (see Section 7).

The fields of the $\mathfrak{h}$ Poisson sigma model are organized in an operation, referred to as the $\mathfrak{h}$ Hamilton de Rham superfield operation below. This is a de Rham superfield realization of the $\mathfrak{h}$ Hamilton equivariant operation of ΠTM and is concretely constructed as follows. Each of the generators x^i, $\tilde{X}^i$, y_i, $\tilde{Y}_i$, γ^a, Γ^a of $\mathrm{Fun}(\Pi T \Pi T^* M)\hat{\otimes} W(\mathfrak{h})$ is realized as a de Rham superfield, denoted by the same symbol. The values $x^i(z,\zeta)$, $\tilde{X}^i(z,\zeta)$, $y_i(z,\zeta)$, $\tilde{Y}_i(z,\zeta)$, $\gamma^a(z,\zeta)$, $\Gamma^a(z,\zeta)$ of these superfields for varying (z,ζ) generate, after imposing a natural smoothness requirement, a formal graded associative algebra $\mathcal{F}(\Sigma, M, \mathfrak{h})$. The $\mathfrak{h}$ Hamilton de Rham superfield operation is the $\mathfrak{h}$ operation over $\mathcal{F}(\Sigma, M, \mathfrak{h})$ whose derivations $j(r)$, $l(r)$, $r \in \mathfrak{h}$, and s are defined in terms of the de Rham superfield generators x^i, $\tilde{X}^i$, y_i, $\tilde{Y}_i$, γ^a, Γ^a of $\mathcal{F}(\Sigma, M, \mathfrak{h})$ according to expressions formally identical to those valid for the corresponding generators of $\mathrm{Fun}(\Pi T \Pi T^* M)\hat{\otimes} W(\mathfrak{h})$, as expounded in Section 8. The derivation d defined in (6.1) and (8.8) is realized as the de Rham differential d, Eq. (9.4), as indicated by the use of the same notation.

A de Rham superfield Λ is Hamilton, if $\Lambda(z,\zeta)$ belongs to $\mathcal{F}(\Sigma, M, \mathfrak{h})$ for all (z,ζ). For any Hamilton de Rham superfield Λ, $\int_\Sigma \mu\Lambda$ is defined and belongs to $\mathcal{F}(\Sigma, M, \mathfrak{h})$.

A Hamilton de Rham superfield Λ is local if $\Lambda(z,\zeta)$ depends only on the values of the de Rham superfield generators x^i, $\tilde{X}^i$, y_i, $\tilde{Y}_i$, γ^a, Γ^a and a finite number of their derivatives at (z,ζ). Clearly, each superfield x^i, $\tilde{X}^i$, y_i, $\tilde{Y}_i$, γ^a, Γ^a is Hamilton and local.

If the Hamilton de Rham superfield Λ is a representative of a mod d $\mathfrak{h}$ Hamilton de Rham superfield basic cohomology class, then $\int_\Sigma \mu\Lambda$ is a representative of an $\mathfrak{h}$ Hamilton de Rham superfield basic cohomology class. Indeed, as $j(r)\Lambda$, $l(r)\Lambda$, $r \in \mathfrak{h}$, and $s\Lambda$ all vanish mod d, $j(r)\int_\Sigma \mu\Lambda$, $l(r)\int_\Sigma \mu\Lambda$, $r \in \mathfrak{h}$, and $s\int_\Sigma \mu\Lambda$, vanish exactly on account of (9.7).

The crucial observations, which we shall exploit extensively below, are the following. *Every element $\mathcal{O}$ of Fun $(\Pi T \Pi T^* M)\hat{\otimes} W(\mathfrak{h})$ yields a local Hamilton de Rham superfield of $\mathcal{F}(\Sigma, M, \mathfrak{h})$, denoted also by $\mathcal{O}$ and called its Hamilton de Rham superfield realization, by substituting each of the generators of* $\mathrm{Fun}(\Pi T \Pi T^* M)\hat{\otimes} W(\mathfrak{h})$ *with the corresponding superfield generator of $\mathcal{F}(\Sigma, M, \mathfrak{h})$. Every relation involving one or more elements in Fun $(\Pi T \Pi T^* M)\hat{\otimes} W(\mathfrak{h})$ entails a formally identical relation involving their Hamilton de Rham superfield realizations in $\mathcal{F}(\Sigma, M, \mathfrak{h})$. In particular, representatives of mod* d $\mathfrak{h}$ *Hamilton equivariant cohomology classes yield directly local Hamilton de Rham superfields representing mod* d $\mathfrak{h}$ *Hamilton de Rham superfield basic cohomology classes.*

The Lagrangian of the $\mathfrak{h}$ Poisson sigma model is derived directly from the following degree 2 element of $\mathrm{Fun}(\Pi T \Pi T^* M)\hat{\otimes} W(\mathfrak{h})$

$$\mathcal{L}_\pi = y_i \tilde{X}^i + \tfrac{1}{2}\pi^{ij}(x) y_i y_j - \Phi_\Gamma(x). \tag{10.1}$$

Here, π^{ij} is a 2-vector satisfying

$$[\pi, \pi] = 0, \quad \text{i.e. } \pi^{il}\partial_l \pi^{jk} + \pi^{jl}\partial_l \pi^{ki} + \pi^{kl}\partial_l \pi^{ij} = 0, \tag{10.2}$$

$$[\pi,\varpi]=0,\quad \text{i.e. } \varpi^{il}\partial_l\pi^{jk}+\varpi^{jl}\partial_l\pi^{ki}+\varpi^{kl}\partial_l\pi^{ij} +\pi^{il}\partial_l\varpi^{jk}+\pi^{jl}\partial_l\varpi^{ki}+\pi^{kl}\partial_l\varpi^{ij}=0. \tag{10.3}$$

We further demand that the Hamilton action ς satisfies

$$[\pi,\varsigma(r)]=0,\quad \text{i.e. } \pi^{ij}\partial_j\varsigma(r)=0 \tag{10.4}$$

with $r\in\mathfrak{h}$. These restrictions on π^{ij} and ς, whose justification will be provided in the next section, have the following simple geometrical interpretation. π^{ij} is another Poisson 2-vector of M compatible with the given Poisson 2-vector ϖ^{ij} (cf. Eq. (3.1)). The $\varsigma(r)$ are Casimir functions of the Poisson structure of π^{ij}. To avoid possible confusion, *below, unless otherwise stated, we tacitly assume that the Poisson structure of M is that defined by the Poisson* 2*-vector* ϖ^{ij}.

Let $\mathrm{Cas}_\pi(M)$ be the space of functions f satisfying

$$[\pi,f]=0,\quad \text{i.e. } \pi^{ij}\partial_j f=0. \tag{10.5}$$

Using (10.3) and the simple relation $\{f,g\}=[f,[\varpi,g]]$, $f,g\in\mathrm{Fun}(M)$, it is easy to show that $\mathrm{Cas}_\pi(M)$ a Poisson subalgebra of $\mathrm{Fun}(M)$, the "π-twisted" Casimir subalgebra.

Using the relation $l_M(u_f)\pi=-[[\varpi,f],\pi]$, $f\in\mathrm{Fun}(M)$, it is simple to check that, for f in $\mathrm{Cas}_\pi(M)$, $l_M(u_f)\pi^{ij}=0$. Hence, the Poisson 2-vector π^{ij} is invariant under the Hamilton vector fields of the π-twisted Casimir functions.

From (10.4) and these simple considerations, it follows that, for $r\in\mathfrak{h}$, $\varsigma(r)\in\mathrm{Cas}_\pi(M)$ and that the Poisson 2-vector π^{ij} is invariant under the Hamilton action ς, i.e. $l_M(u_{\varsigma(r)})\pi^{ij}=0$, for $r\in\mathfrak{h}$.

From here, using (10.3) and proceeding as in Section 7, we find that $\mathcal{L}_\pi$ satisfies

$$\begin{aligned} &j(r)\mathcal{L}_\pi=0,\qquad l(r)\mathcal{L}_\pi=0,\\ &s\mathcal{L}_\pi=\mathrm{d}(y_i\tilde{X}^i-\Phi_\Gamma(x)+\tfrac{1}{2}\pi^{ij}(x)y_iy_j-\tfrac{1}{2}\varpi^{ij}(x)y_iy_j) \end{aligned} \tag{10.6}$$

for any element r of $\mathfrak{h}$. Hence, $\mathcal{L}_\pi$ *is a representative of a mod d degree* 2 $\mathfrak{h}$ *Hamilton equivariant cohomology class*.

The treatment of the $\mathfrak{h}$ Poisson sigma model requires going onto the $\mathfrak{h}$ Hamilton de Rham superfield operation. The Lagrangian of the model is the local Hamilton de Rham superfield realization of $\mathcal{L}_\pi$ and is obtained from (10.1) using (6.1) and (8.8). The action $\mathcal{S}_\pi$ of the model, given as usual by $\int_\Sigma \mu\mathcal{L}_\pi$, thus reads explicitly

$$\mathcal{S}_\pi=\int_\Sigma\mu\left(y_i\,\mathrm{d}x^i+\frac{1}{2}\pi^{ij}(x)y_iy_j-\Phi_{d\gamma}(x)\right). \tag{10.7}$$

So, $\mathcal{S}_\pi$ has degree 0 and, by (10.6) and the above discussion, satisfies

$$j(r)\mathcal{S}_\pi=0,\qquad l(r)\mathcal{S}_\pi=0,\qquad s\mathcal{S}_\pi=0 \tag{10.8}$$

with r in $\mathfrak{h}$. Thus, $\mathcal{S}_\pi$ *is a representative of a degree* 0 $\mathfrak{h}$ *Hamilton de Rham superfield basic cohomology class*.

If $\pi^{ij}=\varpi^{ij}$, $\mathrm{Cas}_\pi(M)=\mathrm{Cas}(M)$ on account of (3.5) and (10.5). Hence, the Poisson subalgebra $\varsigma(\mathfrak{h})$ is contained in the Casimir subalgebra $\mathrm{Cas}(M)$ and, as the Hamilton vector field of a Casimir function vanishes identically by (3.5) and (3.7), the action of the derivations

$j(r)$, $l(r)$ is trivial for all r in $\mathfrak{h}$. In this way, the underlying $\mathfrak{h}$ Hamilton equivariant cohomology of ΠTM reduces to ordinary cohomology and, in this sense, is trivialized. When, conversely, $\pi^{ij} \neq \varpi^{ij}$, $\mathrm{Cas}_\pi(M) \neq \mathrm{Cas}(M)$ in general. Therefore, the above argument does not apply and the action of the derivations $j(r)$, $l(r)$ for r in $\mathfrak{h}$ is generally non-trivial. In this way, the $\mathfrak{h}$ Hamilton equivariant cohomology of ΠTM is generally non-trivial as well. The import of this observation has been discussed at the end of Section 1.

11. The Batalin–Vilkoviski formulation of the $\mathfrak{h}$ Poisson sigma model

The superfield formulation of the Poisson sigma model was developed in order to implement the Batalin–Vilkoviski quantization algorithm [10]. It is encouraging to find out that the action $\mathcal{S}_\pi$ constructed above satisfies the Batalin–Vilkoviski classical master equation.

Here, we shall use the convenient de Rham superfield formalism. We identify the fields and antifields with x^i and y_i, respectively. The Batalin–Vilkoviski odd symplectic form of the Poisson sigma model is

$$\hat{\Omega}_{\mathrm{BV}} = \int_\Sigma \mu \delta x^i \delta y_i. \tag{11.1}$$

Note that there is no term corresponding to γ^a and its antifield in the symplectic form, since these are considered fixed non-dynamical background fields.

Therefore, the Batalin–Vilkoviski antibrackets are given by

$$(x^i(z,\zeta), y_i(z',\zeta')) = \delta^i{}_j \delta(z,\zeta;z',\zeta'), \tag{11.2}$$

where the super delta distribution δ is given by

$$\delta(z,\zeta;z',\zeta') = \tfrac{1}{2}\delta^{0,2}_{\alpha'\beta'}(z;z')\zeta^{\alpha'}\zeta^{\beta'} + \delta^{1,1}_{\alpha\beta'}(z;z')\zeta^{\alpha}\zeta^{\beta'} + \tfrac{1}{2}\delta^{2,0}_{\alpha\beta}(z;z')\zeta^{\alpha}\zeta^{\beta}, \tag{11.3}$$

$\delta^{p,1-p}(z;z')$ being the usual delta distributions for forms on Σ. For a superfield Ψ:

$$\int_\Sigma \mu' \delta(z,\zeta;z',\zeta')\Psi(z',\zeta') = \Psi(z,\zeta). \tag{11.4}$$

Using (11.2) and (11.4), one verifies that

$$(\mathcal{S}_\pi, \mathcal{S}_\pi) = \int_\Sigma \mu \left[2\pi^{ij}(x) y_i \partial_j \Phi_{d\gamma}(x) - \frac{1}{3}(\pi^{il}\partial_l\pi^{jk} + \pi^{jl}\partial_l\pi^{ki} + \pi^{kl}\partial_l\pi^{ij})(x) y_i y_j y_k \right]. \tag{11.5}$$

Hence, *the action $\mathcal{S}_\pi$ satisfies the Batalin–Vilkoviski classical master equation*:

$$(\mathcal{S}_\pi, \mathcal{S}_\pi) = 0, \tag{11.6}$$

if (10.2) *and* (10.4) *hold*. This analysis provides a field theoretic justification of conditions (10.2) and (10.4).

The field equations entailed by the action $\mathcal{S}_\pi$ are

$$\mathrm{d}x^i + \pi^{ij}(x) y_j = 0, \qquad \mathrm{d}y_i + \tfrac{1}{2}\partial_i \pi^{jk}(x) y_j y_k - \partial_i \Phi_{d\gamma}(x) = 0. \tag{11.7}$$

By applying the differential d to both equations, one obtains the integrability condition

$$\pi^{ij}(x)\partial_j\Phi_{d\gamma}(x) - \tfrac{1}{2}(\pi^{il}\partial_l\pi^{jk} + \pi^{jl}\partial_l\pi^{ki} + \pi^{kl}\partial_l\pi^{ij})(x)y_jy_k = 0. \tag{11.8}$$

Hence, *the field equations are solvable if* (10.2) *and* (10.4) *hold*. It is interesting to note that the requirement of integrability of the field equations leads to the same restrictions as those implied by the master equations.

The Batalin–Vilkoviski variation of the superfields x^i, y_i are given by

$$\begin{aligned}\delta_\pi x^i &= (S_\pi, x^i) = \mathrm{d}x^i + \pi^{ij}(x)y_j,\\ \delta_\pi y_i &= (S_\pi, y_i) = \mathrm{d}y_i + \tfrac{1}{2}\partial_i\pi^{jk}(x)y_jy_k - \partial_i\Phi_{d\gamma}(x).\end{aligned} \tag{11.9}$$

As is well known [10], if the master equation (11.6) is fulfilled, δ_π is a degree 1 nilpotent derivation on $\mathcal{F}(\Sigma, M, \mathfrak{h})$

$$\delta_\pi^2 = 0. \tag{11.10}$$

From (6.1) and (8.8), it is easy to see that δ_π is nothing but the Hamilton de Rham superfield realization of a degree 1 derivation w_π on $\mathrm{Fun}(\Pi T\Pi T^*M)\hat{\otimes}W(\mathfrak{h})$ defined by

$$\begin{aligned}&w_\pi x^i = \tilde{X}^i + \pi^{ij}(x)y_j, \qquad w_\pi\tilde{X}^i = -\pi^{ij}(x)\tilde{Y}_j - \partial_j\pi^{ik}(x)\tilde{X}^jy_k,\\ &w_\pi y_i = \tilde{Y}_i + \tfrac{1}{2}\partial_i\pi^{jk}(x)y_jy_k - \partial_i\Phi_\Gamma(x),\\ &w_\pi\tilde{Y}_i = -\tfrac{1}{2}\partial_i\partial_j\pi^{kl}(x)\tilde{X}^jy_ky_l + \partial_i\pi^{jk}(x)y_j\tilde{Y}_k + \partial_i\partial_j\Phi_\Gamma(x)\tilde{X}^j,\\ &w_\pi\gamma^a = 0, \qquad w_\pi\Gamma^a = 0.\end{aligned} \tag{11.11}$$

It is straightforward to verify that

$$\begin{aligned}&[w_\pi, w_\pi] = 0, \qquad [w_\pi, j(r)] = 0, \qquad [w_\pi, l(r)] = 0,\\ &[w_\pi, s] = 0, \qquad [w_\pi, d] = 0\end{aligned} \tag{11.12}$$

for $r \in \mathfrak{h}$, *if* (10.2), (10.3) *and* (10.4) *hold*. Hence, the compatibility of the nilpotent operator w_π and the derivations of the $\mathfrak{h}$ Hamilton equivariant operation of ΠTM leads to condition (10.3) in addition to conditions (10.2) and (10.4) previously obtained.

12. $\mathfrak{h}$ Hamilton de Rham superfield basic cohomology classes and Batalin–Vilkoviski observables

Next, we want to investigate under which conditions local representatives of $\mathfrak{h}$ Hamilton de Rham superfield basic cohomology classes are also Batalin–Vilkoviski observables of the $\mathfrak{h}$ Poisson sigma model, i.e. local representatives of the δ_π cohomology classes [10].

Let $\mathcal{O}$ be a local Hamilton de Rham superfield in $\mathcal{F}(\Sigma, M, \mathfrak{h})$ representing a mod d $\mathfrak{h}$ Hamilton de Rham superfield basic cohomology class. Then, for any singular supercycle Z (cf. Section 10):

$$\langle Z, \mathcal{O}\rangle = \int_Z \mu\mathcal{O} \tag{12.1}$$

is a representative of an $\mathfrak{h}$ Hamilton de Rham superfield basic cohomology class. Indeed, as $j(r)\mathcal{O}$, $l(r)\mathcal{O}$, $r \in \mathfrak{h}$, and $s\mathcal{O}$ all vanish mod d, $j(r)\int_Z \mu\mathcal{O}$, $l(r)\int_Z \mu\mathcal{O}$, $r \in \mathfrak{h}$, and $s\int_Z \mu\mathcal{O}$, vanish exactly on account of (9.13).

According to the Batalin–Vilkoviski theory, $\langle Z, \mathcal{O}\rangle$ is an observable of the $\mathfrak{h}$ Poisson sigma model for all singular supercycles Z, provided

$$\delta_\pi\langle Z, \mathcal{O}\rangle = 0 \tag{12.2}$$

for all such Z [10]. This poses further restriction on $\mathcal{O}$, namely

$$\delta_\pi\mathcal{O} = \mathrm{d}\mathcal{X} \tag{12.3}$$

for some local Hamilton de Rham superfield $\mathcal{X}$ in $\mathcal{F}(\Sigma, M, \mathfrak{h})$.

Assume that $\mathcal{O}$ is the Hamilton de Rham superfield realization of some element of $\mathrm{Fun}(\Pi T\Pi T^*M)\hat{\otimes}W(\mathfrak{h})$, which we also denote by $\mathcal{O}$. Then, on one hand $\mathcal{O}$ must obey (7.1) with $j(u)$, $l(u)$, u^i replaced by $j(r)$, $l(r)$, r with r in $\mathfrak{h}$. On the other, recalling that δ_π is the Hamilton de Rham superfield realization of w_π (cf. Section 11), $\mathcal{O}$ must satisfy the further condition

$$w_\pi\mathcal{O} = \mathrm{d}\mathcal{X} \tag{12.4}$$

for some element $\mathcal{X}$ of $\mathrm{Fun}(\Pi T\Pi T^*M)\hat{\otimes}W(\mathfrak{h})$, in analogy to (12.3).

Suppose $\mathcal{O} = \beta(x, y)$ is of the form (7.2). Then, $\beta(x, y)$ obeys (7.12) with f in $\varsigma(\mathfrak{h})$. Using (11.11), one computes

$$w_\pi\beta(x, y) = \mathrm{d}\beta(x, y) - [\pi, \beta](x, y) + [\Phi_\Gamma, \beta](x, y). \tag{12.5}$$

Therefore, imposing that $\beta(x, y)$ satisfies (12.4), we obtain further conditions:

$$[f, \beta(x, y)] = 0, \qquad q_\pi\beta(x, y) = 0 \tag{12.6}$$

for all f in $\varsigma(\mathfrak{h})$, where q_π is defined by (3.9) with ϖ substituted by π. Note that the first condition (12.6) coincides with the first condition (7.12). When $\pi^{ij} = \varpi^{ij}$, (12.6) reduces to (7.12) and no further restriction is implied by (12.4). In general, imposing (7.12) and (12.6) simultaneously is rather restrictive and only trivial solutions of these conditions are available.

Suppose $\mathcal{O} = \sigma(x, \tilde{X})$ is of the form (7.6). Then, $\sigma(x, \tilde{X})$ obeys (7.16). Using (11.11), one computes

$$w_\pi\sigma(x, \tilde{X}) = \mathrm{d}(\sigma(x, \tilde{X}) - k_\pi\sigma(x, \tilde{X})) + k_\pi d_M\sigma(x, \tilde{X}), \tag{12.7}$$

where k_π is the degree 0 derivation defined by (7.8) with ϖ substituted by π. Therefore, imposing that $\sigma(x, \tilde{X})$ satisfies (12.4), we get further condition:

$$k_\pi d_M\sigma(x, \tilde{X}) = 0. \tag{12.8}$$

When $\pi^{ij} = \varpi^{ij}$, (12.8) reduces to (7.16) and no further restriction is implied by (12.4). (7.16) and (12.8) are simultaneously solved by all closed p-forms $\sigma_{i_1\cdots i_p}$ of M. However, non-trivial observables are yielded only for $p = 0, 1, 2$.

13. Discussion and examples

In this final section, we illustrate the formal analysis worked out above by providing a few examples of manifolds M endowed with a pair of 2-vectors ϖ^{ij}, π^{ij} satisfying (3.1), (10.2) and (10.3). For convenience, we write the 2-vector π^{ij} as

$$\pi^{ij} = \varpi^{ij} + \vartheta^{ij}, \tag{13.1}$$

where ϑ^{ij} is a 2-vector satisfying (10.2) and (10.3) with π^{ij} replaced by ϑ^{ij}. Hamilton actions ς of a finite-dimensional Lie algebra $\mathfrak{h}$ on M satisfying (10.4) are most efficiently constructed as follows. One chooses a finite set of linearly independent functions of $\mathrm{Cas}_\pi(M)$ and defines $\mathfrak{h}$ to be the Lie algebra spanned by these functions under Poisson brackets, so that ς becomes simply the identity map. In what follows, we follow closely the methodology of Ref. [19].

13.1. Two-dimensional Poisson spaces

Let M be a two-dimensional manifold. We equip M with an auxiliary metric g_{ij}. Any 2-vector ζ^{ij} can be written as

$$\zeta^{ij} = \epsilon^{ij}\alpha \tag{13.2}$$

for some function α, where ϵ^{ij} is the Levi–Civita 2-vector associated to g_{ij}. Let ϖ^{ij}, ϑ^{ij} be two 2-vectors and let μ, ν be the corresponding functions in the representation (13.2). Then ϖ^{ij}, ϑ^{ij} automatically are Schouten commuting Poisson 2-vectors, irrespective of the specific form of μ, ν. A function f on M belongs to $\mathrm{Cas}_\pi(M)$, if and only if

$$(\mu + \nu)\partial_i f = 0. \tag{13.3}$$

So, f is constant in the open subsets of M where the sum $\mu+\nu$ is non-vanishing and arbitrary in the open subsets of M where $\mu + \nu$ vanishes. Since $\{f, g\} = \mu\epsilon^{ij}\partial_i f\partial_j g$, $\mathrm{Cas}_\pi(M)$ is a generally non-Abelian Poisson subalgebra of $\mathrm{Fun}(M)$.

13.2. Three-dimensional Poisson spaces

Let M be a three-dimensional manifold. We equip M with an auxiliary metric g_{ij}. Any 2-vector ζ^{ij} can be written as

$$\zeta^{ij} = \epsilon^{ijk}\alpha_k \tag{13.4}$$

for some 1-form α_i, where ϵ^{ijk} is the Levi–Civita 3-vector associated to g_{ij}. Let ϖ^{ij}, ϑ^{ij} be two 2-vectors and let μ_i, ν_i be the corresponding 1-forms in the representation (13.4). Then, ϖ^{ij}, ϑ^{ij} are Schouten commuting Poisson 2-vectors, if and only if

$$\epsilon^{ijk}\mu_i\nabla_j\mu_k = 0, \qquad \epsilon^{ijk}(\mu_i\nabla_j\nu_k + \nu_i\nabla_j\mu_k) = 0, \qquad \epsilon^{ijk}\nu_i\nabla_j\nu_k = 0, \tag{13.5}$$

where ∇_i is the Riemannian connection of g_{ij}. It is known that the first and third condition have the local solution

$$\mu_i = u\partial_i p, \qquad \nu_i = v\partial_i q, \tag{13.6}$$

where u, v, p, q are certain local functions [19]. The remaining condition can then be cast as

$$uv\epsilon^{ijk}\partial_i \ln\left(\frac{u}{v}\right)\partial_j p\partial_k q = 0. \tag{13.7}$$

A function f on M belongs to $\mathrm{Cas}_\pi(M)$, if and only if

$$\epsilon^{ijk}(\mu+\nu)_j\partial_k f = 0, \tag{13.8}$$

or, on account of (13.6):

$$\epsilon^{ijk}(u\partial_j p + v\partial_j q)\partial_k f = 0. \tag{13.9}$$

By (13.8), if the 1-form $\mu+\nu$ vanishes at most in the complement of an open dense set, then, at least locally, $\partial_i f = k_f(\mu+\nu)_i$ for some function k_f. In that case, as it easy to see from the relation $\{f, g\} = \epsilon^{ijk}\mu_i\partial_j f\partial_k g$, $\mathrm{Cas}_\pi(M)$ is an Abelian Poisson subalgebra of $\mathrm{Fun}(M)$. For instance, one may consider $M = \mathbb{R}^3$ equipped with the Schouten commuting Poisson 2-vectors ϖ^{ij}, ϑ^{ij} corresponding to the compatible Poisson structures:

$$\{x_1, x_2\} = x_3, \qquad \{x_2, x_3\} = x_1, \qquad \{x_3, x_1\} = x_2, \tag{13.10}$$

$$\{x_1, x_2\}_\vartheta = \tfrac{1}{2} - (x_3 + \tfrac{1}{2})^2, \qquad \{x_2, x_3\}_\vartheta = 0, \qquad \{x_3, x_1\}_\vartheta = 0. \tag{13.11}$$

The resulting Poisson 2-vector π^{ij} appears in the Poisson sigma model describing two-dimensional Euclidean R^2 gravity with cosmological constant [4]. A solution of Eq. (13.8) is

$$f(x_1, x_2, x_3) = \tfrac{1}{2}(x_1^2 + x_2^2) - \tfrac{1}{3}x_3(x_3^2 - \tfrac{3}{4}). \tag{13.12}$$

As another example, one may consider $M = \mathbb{R}^2 \times \mathbb{S}^1$ with the Schouten commuting Poisson 2-vectors ϖ^{ij}, ϑ^{ij} defined by the compatible Poisson structures:

$$\{x_1, x_2\} = 0, \qquad \{x_1, \varphi\} = 0, \qquad \{x_2, \varphi\} = P(x_1, x_2), \tag{13.13}$$

$$\{x_1, x_2\}_\vartheta = 0, \qquad \{x_1, \varphi\}_\vartheta = -Q(x_1, x_2), \qquad \{x_2, \varphi\}_\vartheta = 0, \tag{13.14}$$

where $P(x_1, x_2)$, $Q(x_1, x_2)$ are certain functions In this case, Eq. (13.8) reduces to

$$P(x_1, x_2)\partial_{x_2} f - Q(x_1, x_2)\partial_{x_1} f = 0, \quad P(x_1, x_2)\partial_\varphi f = Q(x_1, x_2)\partial_\varphi f = 0. \tag{13.15}$$

In the generic situation, $\partial_\varphi f = 0$ and the first equation can be treated with standard analytical techniques.

13.3. Four-dimensional Poisson spaces

Let M be a four-dimensional manifold. We equip M with an auxiliary metric g_{ij}. Any 2-vector ζ^{ij} can be written as

$$\zeta^{ij} = \tfrac{1}{2}\epsilon^{ijkl}\alpha_{kl} \tag{13.16}$$

for some 2-form α_{ij}, where ϵ^{ijkl} is the Levi–Civita 4-vector associated to g_{ij}. Let ϖ^{ij}, ϑ^{ij} be two 2-vectors and let μ_{ij}, ν_{ij} be the corresponding 2-forms in the representation (13.16).

Then, ϖ^{ij}, ϑ^{ij} are Schouten commuting Poisson 2-vectors, if and only if

$$\epsilon^{jklm}\mu_{jk}\nabla_l\mu_{mi} = 0, \qquad \epsilon^{jklm}(\mu_{jk}\nabla_l\nu_{mi} + \nu_{jk}\nabla_l\mu_{mi}) = 0, \qquad \epsilon^{jklm}\nu_{jk}\nabla_l\nu_{mi} = 0, \tag{13.17}$$

where again ∇_i is the Riemannian connection of g_{ij}. If one restricts oneself to degenerate Poisson 2-vectors, i.e. with everywhere vanishing determinant, it is known that the first and third condition have the local solution:

$$\mu_{ij} = u(\partial_i p\partial_j q - \partial_j p\partial_i q), \qquad \nu_{ij} = v(\partial_i r\partial_j s - \partial_j r\partial_i s), \tag{13.18}$$

where u, v, p, q, r, s are certain local functions [19]. The remaining condition can then be cast as

$$\begin{aligned}&\epsilon^{jklm}[u\partial_j p\partial_k q(\partial_m r\nabla_l(v\partial_i s) - \partial_m s\nabla_l(v\partial_i r))\\ &\quad + v\partial_j r\partial_k s(\partial_m p\nabla_l(u\partial_i q) - \partial_m q\nabla_l(u\partial_i p))] = 0.\end{aligned} \tag{13.19}$$

A function f on M belongs to $\mathrm{Cas}_\pi(M)$, if and only if

$$\epsilon^{ijkl}(\mu + \nu)_{jk}\partial_l f = 0, \tag{13.20}$$

or, on account of (13.18):

$$\epsilon^{ijkl}(u\partial_j p\partial_k q + v\partial_j r\partial_k s)\partial_l f = 0. \tag{13.21}$$

There is not much that can be said in general on the solution of this equation. As an example, one can consider $M = \mathbb{R}^3 \times \mathbb{R}$ equipped with the Schouten commuting Poisson 2-vectors ϖ^{ij}, ϑ^{ij} corresponding to the compatible Poisson structures:

$$\{x_i, x_j\} = \sum_{k=1}^{3}\varepsilon_{ijk}x_k y, \qquad \{x_i, y\} = 0, \tag{13.22}$$

$$\{x_i, x_j\}_\vartheta = 0, \qquad \{x_i, y\}_\vartheta = \sum_{j,k=1}^{3}\varepsilon_{ijk}(a_j - a_k)x_j x_k, \tag{13.23}$$

where ε_{ijk} is three-dimensional totally antisymmetric symbol and the a_i the real numbers. The resulting Poisson 2-vector π^{ij} is that of the famous Sklyanin Poisson structure [20]. Eq. (13.20) is solved by

$$f_1(x_1, x_2, x_3, y) = \frac{1}{2}\sum_{i=1}^{3}a_i x_i^2 - \frac{1}{4}y^2, \qquad f_2(x_1, x_2, x_3, y) = \frac{1}{2}\sum_{i=1}^{3}x_i^2. \tag{13.24}$$

f_2 is a common Casimir function of both ϖ^{ij} and ϑ^{ij} and so is not of any use.

13.4. Affine Lie–Poisson spaces

Lie–Poisson spaces appear in the Poisson sigma model describing two-dimensional Yang–Mills theory [4]. An interesting generalization is provided by the so-called affine Lie–Poisson spaces [17]. As an example, we consider $M = \mathbb{R}^n$ with the 2-vectors

$$\varpi^{ij}(x) = c_k^{ij}x^k, \qquad \vartheta^{ij}(x) = a^{ij}, \tag{13.25}$$

where the constants c_k^{ij}, a^{ij} satisfy

$$c_m^{ij}c_l^{mk} + c_m^{jk}c_l^{mi} + c_m^{ki}c_l^{mj} = 0, \tag{13.26}$$

$$c_m^{ij}a^{mk} + c_m^{jk}a^{mi} + c_m^{ki}a^{mj} = 0. \tag{13.27}$$

As is well-known, (13.26) and (13.27) state that $\mathbb{R}^{n\vee}$ is a Lie algebra with structure constants c_k^{ij} and that a^{ij} is a Chevalley–Eilenberg 2-cocycle of $\mathbb{R}^{n\vee}$. ϖ^{ij}, ϑ^{ij} are Schouten commuting Poisson 2-vectors. ϖ^{ij} is usually called Kirillov–Kostant–Souriau Poisson structure [21–23]. A function f on M belongs to $\mathrm{Cas}_\pi(M)$, if and only if

$$(c_k^{ij}x^k + a^{ij})\partial_j f = 0. \tag{13.28}$$

An example is provided by $M = \mathbb{R}^4$ with the Poisson structures defined by

$$\{x_0, x_i\} = x_{i+1}, \quad 1 \le i \le 3, \qquad \{x_i, x_j\} = 0, \quad 1 \le i, j \le 3, \tag{13.29}$$

$$\{x_0, x_i\}_\vartheta = a\delta_{i,1}, \quad 1 \le i \le 3, \qquad \{x_i, x_j\}_\vartheta = 0, \quad 1 \le i, j \le 3, \tag{13.30}$$

where a is a real number and $x_4 = 0$ by convention. A solution of (13.28) is given by

$$f(x_0, x_1, x_2, x_3) = (x_1x_3 - \tfrac{1}{2}(x_2 + a)^2)g(x_3), \tag{13.31}$$

where g is an arbitrary function.

13.5. Compact Poisson Riemannian symmetric spaces

Let M be a compact Riemannian symmetric space with metric g_{ij}. Then, if σ_{ij}, τ_{ij} are two harmonic 2-forms:

$$\varpi^{ij} = g^{ik}g^{jl}\sigma_{kl}, \qquad \vartheta^{ij} = g^{ik}g^{jl}\tau_{kl} \tag{13.32}$$

are Schouten commuting Poisson 2-vectors [17]. Of course, this example is to be considered trivial unless the Betti number $b_2(M) \ge 2$.

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References

[1] N. Ikeda, Two-dimensional gravity and nonlinear gauge theory, Ann. Phys. 235 (1994) 435. arXiv:hep-th/9312059.
[2] P. Schaller, T. Strobl, Poisson structure induced (topological) field theories, Mod. Phys. Lett. A 9 (1994) 3129. arXiv:hep-th/9405110.
[3] P. Schaller, T. Strobl, Poisson sigma models: a generalization of 2-d gravity Yang–Mills systems. arXiv:hep-th/9411163.
[4] P. Schaller, T. Strobl, A brief introduction to Poisson sigma-models. arXiv:hep-th/9507020.

[5] M. Kontsevich, Deformation quantization of Poisson manifolds. arXiv:q-alg/9709040.
[6] A.S. Cattaneo, G. Felder, A path integral approach to the Kontsevich quantization formula, Commun. Math. Phys. 212 (2000) 591. arXiv:math.qa/9902090.
[7] A.S. Cattaneo, G. Felder, Poisson sigma models and deformation quantization, Mod. Phys. Lett. A 16 (2001) 179. arXiv:hep-th/0102208.
[8] A.S. Cattaneo, G. Felder, On the AKSZ formulation of the Poisson sigma model, Lett. Math. Phys. 56 (2001) 163. arXiv:math.qa/0102108.
[9] L. Baulieu, A.S. Losev, N.A. Nekrasov, Target space symmetries in topological theories, I. arXiv:hep-th/0106042.
[10] J. Gomis, J. Paris, S. Samuel, Antibracket, antifields and gauge theory quantization, Phys. Rep. 259 (1995) 1. arXiv:hep-th/9412228.
[11] S. Cordes, G.W. Moore, S. Ramgoolam, Lectures on 2-d Yang–Mills theory, equivariant cohomology and topological field theories, Nucl. Phys. Proc. Suppl. 41 (1995) 184. arXiv:hep-th/9411210.
[12] D. Birmingham, M. Blau, M. Rakowski, G. Thompson, Topological field theory, Phys. Rep. 209 (1991) 129.
[13] S. Ouvry, R. Stora, P. van Baal, On the algebraic characterization of witten topological Yang–Mills theory, Phys. Lett. B 220 (1989) 159.
[14] V. Mathai, D. Quillen, Superconnections, Thom classes and equivariant differential forms, Topology 25 (1986) 85.
[15] M.F. Atiyah, L. Jeffrey, Topological Lagrangians and cohomology, J. Geom. Phys. 7 (1990) 119.
[16] W. Greub, S. Halperin, R. Vanstone, Connections, Curvature and Cohomology, vols. II and III, Academic Press, New York, 1973.
[17] I. Vaisman, Lectures on the Geometry of Poisson Manifolds, Progress in Mathematics, vol. 118, Birkhauser, Basel, 1994.
[18] E. Witten, Topological quantum field theory, Commun. Math. Phys. 117 (1988) 353.
[19] J. Grabowski, G. Marmo, A. Perelomov, Poisson structures: toward a classification, Mod. Phys. Lett. A 8 (1993) 1719.
[20] E.K. Sklyanin, Some algebraic structures connected with the Yang–Baxter equation, Funct. Anal. Appl. 16 (1982) 263 (Funkt. Anal. Pril. 16N4 (1982) 27).
[21] A.A. Kirillov, Unitary representations of nilpotent Lie groups, Russ. Math. Surv. 17 (1962) 53.
[22] B. Kostant, Orbits, symplectic structures and representation theory, in: Proceedings of the US–Japan Seminar of Differential Geometry, Kyoto, 1966, p. 77.
[23] J.-M. Souriau, Structure des Systèmes Dynamiques, Dunod, Paris, 1970.